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The Turning Point Problem in the Asymptotic Integration of Ordinary Linear Differential Equations

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THE TURNING POINT PROBLEM IN THE ASYMPTOTIC INTEGRATION OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

by

Herbert Kurss

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SUMMARY

This report explores three approaches to the turning point problem for linear equations.

The first approach exhibits a full set of solutions of the equation as contour integrals. This is illustrated by the triangular system and the hypergeometric-type equation.

The second approach regards the system as being a perturbation of a comparison equation. In particular, if the leading matrix of the comparison equation is diagonalizable, the rapid convergence of the Picard iterants is shown by an elaboration of a scheme due to K. O. Friedrichs.

The third approach regards the system as being a composite system resulting from the amalgamation of component solvable systems. Particular attention is given to the reduction attainable due to a non-coalescing eigenvalue.

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I. Introduction

The turning point problem has received only sparse treatment for the general linear equation. However, some of the approaches employed for special equations appear to be readily extensible to more general situations. It is the intent of this report to indicate some of the possibilities in this direction which were explored during the past year. The work is still in its preliminary stage so that the discussion is suggestive rather than definitive.

The turning point problem and the related irregular singular point problem is formulated in section II. This is followed by a discussion in section III of special equations for which the turning point problem is directly solvable by contour integrals. The first such equation is the often neglected triangular system (where the equation for each component of a vector solution is decoupled from all the preceding components). The other solvable equation considered at length is, in essence, the equation completely solvable by the generalized hypergeometric functions. The discussion of this equation is based upon the work of H. Scheffé [1]* and H. L. Turrittin [1]. Incidentally, both of the above equation types are solvable even if the turning point occurs at a regular singularity.

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* This notation refers to the first paper listed under the name of Scheffé in the bibliography at the end of this report.

When the equation is not directly solvable one attempts a perturbation scheme which goes back to Liouville and Steckloff. This scheme regards the differential equation as being a perturbation of a comparison equation. The differential equation is then converted into a Volterra integral equation whose inhomogeneous term is the solution of the comparison equation. What remains is to demonstrate the rapid convergence of the iterant solution of the integral equation. One such demonstration is due to K. O. Friedrichs [1, p. 191] and this, denoted as the strong perturbation approach, is elaborated in section IV. Another demonstration stems from R. E. Langer [1] and a possible extension of this weak perturbation is sketched in section V. Finally, one may regard a system as being a composite system, resulting from the amalgamation of component solvable systems. This is indicated in section VI with particular attention to a system in which an eigenvalue does not coalesce at the turning point (Langer [2]).

Whatever merit this report may possess is due to the valuable information, advice, criticism, and encouragement which was unstintingly given by Professor K. O. Friedrichs.

Notation

Throughout the report the following notation is maintained:

$'$ denotes differentiation with respect to x

x is the complex independent variable

$x = 0$ is the turning point

$\rho = \varepsilon^{-1}$ is a large parameter (taken real for convenience)

A, or any other capital letter in an equation is a square matrix

II. The Turning Point Problem

Consider the system of first order equations for the matrix Y

$$Y' = A(x,\varepsilon)Y$$

where ε is a parameter. If A is a 'suitable' function of x and ε then it is well known that as $\varepsilon \to 0$, there exists a full complement of formal solutions $\tilde{Y}(x,\varepsilon)$ (Noaillon [1], Turrittin [2]). Furthermore, in a certain 'minimum sector' of the complex x-plane, bounded by two 'transition lines' which meet at a 'turning point', there exists an actual matrix solution which is 'asymptotic' to the formal solution (Birkhoff [1], Birkhoff-Langer [1], Trjitzinsky [2], Hukuhara [1], Wasow [1, chap. V]). Moreover, this actual solution can be shown to preserve its asymptotic form beyond this 'minimum sector' up to a certain 'maximum sector' which includes several transition lines in its interior (Friedrichs [1, p. 155]. If, as is usual, this maximum sector is the sector of largest extent for the given asymptotic form then it is denoted as the Stokes sector and its bounding lines are called Stokes lines (Stokes [1]). Of course, in a decoupled system the Stokes sectors are in general larger than the maximum sectors. In any event, the neighborhood of the turning point can be covered by a set of overlapping maximum sectors, say $\mathcal{S}_i$, in each of which there is an actual solution, say Y_i, such that

$$Y_i \sim \tilde{Y} \quad \text{in } \mathcal{S}_i \ .$$

The problem is to find the asymptotic behavior of each Y_i in all the other sectors.

To solve this problem it is sufficient to find the asymptotic form (with a remainder the order of a power of ε) of each Y_i at the turning point with the proviso that the columns of this asymptotic form are independent.

One then completes the argument by noting that due to the linearity of the equation, any two fundamental solutions, say Y_h and Y_k, can be related through the mediation of an arbitrary ordinary point x_o by the equation

$$Y_h(x)Y_h^{-1}(x_o) = Y_k(x)Y_k^{-1}(x_o) \ ,$$

i.e.,

$$Y_h(x) = Y_k(x)K(\varepsilon) \quad \text{where} \quad K(\varepsilon) = Y_k^{-1}(x_o)Y_h(x_o) \ .$$

In particular, if x_o is the turning point and Y_h and Y_k have each been 'connected' to the turning point, then $K(\varepsilon)$ is known asymptotically with an error term which is of the order of a power of ε. Consequently, the asymptotic form of Y_h has been 'continued' into the sector $\mathcal{S}_k$, i.e.,

$$Y_h \sim \tilde{Y} \quad \text{in } \mathcal{S}_h$$

$$\sim \tilde{Y}K \quad \text{in } \mathcal{S}_k \ .$$

It is illuminating to compare the above discussion with the following sketch of the parameter free case. As x approaches a non-simple pole of A, i.e., as x approaches an irregular singular point of the equation, there exists a full complement of formal solutions (Fabry [1], Thomé [1], Turrittin [3]). The asymptotic nature of the formal solutions can be shown in a minimum sector (Trjitzinsky [1]) or in a maximum sector (Wasow [1, chap. III], Friedrichs [1, p. 119]) or, for normal equations, in a Stokes sector (Poincaré [1], [2], Birkhoff [2]. (Of course, here the Stokes lines are limiting directions rather than finite lines.) In order to continue asymptotic forms into successive sectors it is sufficient to 'connect' the solution types of <u>any</u> ordinary point of the equation (Turrittin [1]). Finally, one should keep in mind that as $\varepsilon \to 0$ one can map a domain which approaches the irregular singular point onto a finite neighborhood of the turning point. Thereby one solves a special class of turning point problems in terms of a solvable irregular singular point problem. Actually, the irregular singular point is taken to be at $x = \infty$ so that one must stretch the variable of the parameter problem.

III. The Contour Integral Approach

A. The Triangular System

If A is a triangular matrix, i.e., if all the elements of A below the main diagonal are zero, then the solution to

$$Y' = A(x,\varepsilon)Y$$

can be given explicitly. More specifically, Y will be a triangular matrix with exponentials along the main diagonal, exponential integrals along the first superdiagonal, and r-tuple iterated exponential integrals along the r+1 superdiagonal. The exponential integrals are readily evaluated in a sector away from the turning point by what amounts to the saddle point method. In a similar manner one obtains the connection to the turning point and thereby the continuation into successive sectors. Alternately, the continuation problem can be solved directly (Friedrichs [1, p. B-7]). Namely, if a path of an exponential integral is from a_1 to x for x in a sector $\mathcal{S}_1$ then when x meanders into sector $\mathcal{S}_2$ one rewrites the path as a_1 to a_2 and then a_2 to x. The integration from a_1 to a_2 can then be recognized as asymptotic to a variant of the gamma function. The exponential integral is thus continued asymptotically into successive sectors. The singly iterated exponential integral can then be asymptotically integrated with the aid of the above information on the exponential integral, and so on.

A point to emphasize is that the continuation can proceed even when a singularity is present at the turning point. For example,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \left\{ \rho \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} 0 & x^\gamma \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

has $x = 0$ as a turning point but this is an ordinary point only if γ is a non-negative integer. Nevertheless, the continuation problem can be solved in the manner indicated above for all γ.

An attractive feature of the triangular system is the ease with which examples and counterexamples can be displayed. For example, the system

$$\varepsilon \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

illustrates a case where the canonical form of the leading matrix alters abruptly at $x = 0$ and yet a turning point does <u>not</u> occur at $x = 0$.

B. The Hypergeometric-Type Equation

An equation with a parameter which is reduced to a parameter free equation by a simple stretching of the variable must have the following form:

$$\left[\varepsilon\, p_1 \left(x \frac{d}{dx} \right) - x^p p_2 \left(x \frac{d}{dx} \right) \right] y = 0$$

where p_1 and p_2 are polynomials with constant coefficients. The origin of the x-plane is a turning point provided that

$$n_1 \equiv \text{degree of } p_1 \geq \text{degree of } p_2 \equiv n_2 \quad .$$

(The characteristic or eigenvalue equation then has n_2 roots identically zero and $n_1 - n_2$ roots uniformly distributed about the turning point and which coalesce to zero at the turning point.)

In terms of the new (stretched) variable t, where $\varepsilon t^p = x^p$, the equation becomes

$$\left[p_1(t\,\frac{d}{dt}) - t^p p_2(t\,\frac{d}{dt})\right] y = 0 \quad .$$

Attention will now be focused on solving the continuation problem at the irregular singular point $t = \infty$ since this is sufficient to solve the continuation about the turning point at $x = 0$. There are now two decisive points.

Firstly, when this parameter free equations has an ordinary point or regular singularity at $t = 0$, then the non polynomial solutions are simply expressible in terms of generalized hypergeometric functions. Secondly, the asymptotic behavior of the generalized hypergeometric functions is sufficiently well known to permit the solution of the continuation problem around $t = \infty$. Further details on the second point are in the paper by Turrittin [1] and only the first point will be amplified here.

Fuchsian theory assures one that $t = 0$ is a regular singularity provided

$$n_1 \leq n_2 + p \quad .$$

Moreover, if none of the roots of p_1 differ by a multiple of p then there exists a full complement of solutions of the form

$$y = t^{\sigma} \sum_{\ell=0}^{\infty} a_{\ell} t^{p\ell} \quad , \quad a_o \neq 0$$

where σ is a root of the polynomial p_1. The coefficients satisfy the difference equation

$$p_{n_1}(\ell + p\ell + p)a_{\ell+1} - p_{n_2}(\sigma + p\ell)a_{\ell} = 0 \quad .$$

Now one need merely note that this is a two term or binomial difference equation with polynomial coefficients and this in turn can be explicitly solved in terms of factorials. More precisely, let the polynomials be factored in the form

$$a_{\ell+1} \prod_{i=1}^{n_1} (\ell + \beta_i) = a_{\ell} \gamma \prod_{i=1}^{n_2} (\ell + \alpha_i) \quad .$$

Then

$$a_{\ell} \prod_{k=1}^{\ell} \prod_{i=1}^{n_1} (k + \beta_i) = a_o \gamma^{\ell} \prod_{k=1}^{\ell} \prod_{i=1}^{n_2} (\ell + \alpha_i)$$

which can be rewritten as

$$a_{\ell} \prod_{i=1}^{n_1} (\beta_i + \ell)!/\beta_i! = a_o \gamma^{\ell} \prod_{i=1}^{n_2} (\alpha_i + \ell)!/\alpha_i! \quad .$$

Thus in the standard notation for the generalized hypergeometric function

$$y = t^{\sigma}\ {}_{n_2+1}F_{n_1}\left[\begin{matrix} 1,\ \alpha_i+1;\ \gamma t^p \\ \beta_i+1 \end{matrix}\right] .$$

Finally, there is the useful observation that the solutions of

$$\left\{p_{n_1}(td/dt) + t^p\ p_{n_2}(td/dt)\right\} y = 0$$

are contained among the solutions to

$$(td/dt)^k\left\{p_{n_1}(td/dt) + t^p\ p_{n_2}(td/dt)\right\} y = 0 .$$

Thus complete information with regard to the former equation yields partial information with regard to the latter equation and this with little effort.

C. Miscellaneous Equations

A host of irregular singular point problems can be and have been solved by the contour integral approach. This approach appeared in the preceding sections but will now be depicted in a more general version. The approach requires the solutions to have the form of a specific type of integral (Mellin integral, Laplace integral, Euler integral, Jordan-Pockhammer integral, Sommerfeld integral, etc.). This necessitates firstly a restricted path of integration and secondly an integrand which satisfies a transformed equation which is either a difference equation (for the Mellin integral) or a differential equation (for the

others). Moreover, a sufficient number of paths must be available so that a complete set of solutions exist, each in the form of the postulated integral. Finally, the behavior of the solution of the transformed equation must be known explicitly in the vicinity of each of its singularities. Thus the customary transformed equations are the two-term difference equation (Scheffé [1], Turrittin [1]), the first order differential equation (Wasow [2]), and the differential equations of the 'classical' functions (Linstone [1], Wright [1]).

When the above contour integral approach is not directly applicable to an equation it may become applicable after a coordinate transformation. For example, the Mellin transform of the Weber equation (Ince [1, p. 159])

$$y'' + (n + \frac{1}{2} - \frac{1}{4} x^2)y = 0$$

results in a trinomial difference equation. However, the change of dependent variables

$$v = e^{\frac{1}{4}x^2} y$$

results in

$$v'' - xv' + nv = 0$$

which is of Hypergeometric-type.

IV. Strong Perturbations

A. The Diagonal Leading Matrix

1. A Solution of Distinct Exponential Type

This section is concerned with the vector equation

$$u' = (\rho A + B)u$$

in a finite neighborhood of $x = 0$, $\eta(x)$, where the coefficients are restricted by

Condition A: A is diagonal

Condition B: B is uniformly bounded in $\eta(x)$ as $\rho \to \infty$.

(For simplicity B is taken as independent of ρ).

The equation can be rewritten as

$$u' = (\rho A + D + \hat{B})u$$

where

$$B \equiv D + \hat{B} .$$

The subsequent analysis for a particular solution requires B to have a particular term on its main diagonal identically zero. To simplify the discussion let D be chosen as a diagonal matrix with the same diagonal elements as B. Thus all the diagonal elements of $\hat{B}$ are now zero. The solution of

the comparison equation

$$G' = (\rho A + D)G$$

is simply the exponential of a diagonal matrix. Moreover the original equation is now equivalent to the integral equation

$$u(x) = g(x) + \int^{x} G(x)\, G^{-1}(\alpha)\, \hat{B}(\alpha)\, u(\alpha)\, d\alpha$$

where g is a particular solution and G is a fundamental solution of the comparison equation. For definiteness let g be chosen as the second column of G, i.e.,

$$g_i \equiv 0 \qquad i \neq 2$$

$$= \exp \int_{0}^{x} (\rho\lambda_2 + b_{22})dt\,, \qquad i = 2$$

where λ_2 is the appropriate eigenvalue of A. Let

$$v(x) = u(x)\, \exp(-\rho \int_{0}^{x} \lambda_2(t)dt)$$

The integral equation for v is

$$v_i(x) = \sum \int^{x} [\exp \rho \int_{a}^{x} (\lambda_i - \lambda_2)dt]\, \hat{b}_{ij}\, v_j\, da \; ; \quad i \neq 2$$

$$= \exp \int_{o}^{x} b_{22}dt + \sum \int^{x} \hat{b}_{ij}\, v_j\, da \; , \quad i = 2$$

or, in symbolic notation

$$v = \omega + Fv \quad .$$

(Provisionally, the presumption is that $\lambda_i \neq \lambda_2$ for $i \neq 2$, i.e., there is but a single formal expansion of exponential type 2). Paths for the integral equation are chosen in a sector such that the exponential terms in the integrand are bounded (by unity), as $\rho \to \infty$, uniformly for x in the sector and a along the path. Thus in a sector where the asymptotic solution of type 2 is completely recessive all paths can be taken the same. (For details see Friedrichs [1,p.155]). Of course, since the integral equation is of the Volterra type the solution for all finite ρ can be exhibited as the infinite sum of the successive iterations of the integral operator applied to the inhomogeneous term, i. e.,

$$u = \sum_{r=o}^{\infty} F^r(\omega) \quad .$$

Now the key fact is that the iterant solution of this equation

forms an asymptotic series for all x in the sector including $x = 0$. In the successive iterations it is the presence of the exponential integrals which provides the rapid convergence of the v_i, $i \neq 2$, according to positive powers of ϵ.

As a typical exponential integral consider

$$\int_{x}^{x_o} \exp \rho(x^k - t^k)\, dt$$

in the sector where the integrand is uniformly bounded. The point $|x_i| = \epsilon^{1/k}$ is called an intermediate point since the exponential can be regarded as highly damped or very oscillatory only for $|x|$ greater than $|x_i|$. For this portion of the path one can integrate by parts and obtain an $\epsilon/|x|^{k-1}$ dependence which vanishes at least as rapidly as $\epsilon^{1/k}$. Along the path from $x = 0$ to $x = x_i$ the exponential is bounded while the path length vanishes at least as rapidly as $\epsilon^{1/k}$. Hence

$$\int_{x}^{x_o} \exp \rho(x^k - t^k)dt = O(\epsilon^{1/k})$$

for all x in the sector. Finally, one demonstrates the asymptotic convergence of the v_2 in successive iterants by modifying the norm of v. In this modified norm ω is bounded, the v_2 converge, and the convergence of the

remaining v_i is altered but not spoiled. Roughly, if

$$\int^{x} dx \exp \rho \int_{a}^{x} (\lambda_i - \lambda_2)dt = O(\varepsilon^{1/k_i})$$

then one defines $\theta = \frac{1}{2}$ Min $(1/k_i)$ and the modified norm

$$|v| = \rho^{\theta} \sum_{i \neq 2} |v_i| + |v_2|$$

where $|v_i|$ are the maximum norms and concludes that in this norm the Picard iterants converge as ε^{θ} (Friedrichs [1, p. 194]).

As a consequence of the above key fact one obtains the connection to the turning point of a particular solution by successive Picard iterants. (Incidentally, in order that there be no exponential ρ dependence at $x = 0$ it was necessary that the eigenvalue was integrated from the turning point to x). One then adduces the supplemental fact that this particular solution has, when x is a finite distance from the turning point, precisely the desired asymptotic form proceeding in increasing integral powers of $\varepsilon = \rho^{-1}$. This fact is clear from the preceding discussion of the exponential integral where the integral powers of ε are obtained by integration by parts. (Of course, it takes on the order of $1/\theta$ iterations to obtain each successive term. However, the customary formal procedure already presents a canonical form for the asymptotic expansion).

2. A Solution of Non-Distinct Exponential Type

Suppose one starts again with

$$u' = (\rho A + B)u$$

restricted by Conditions A and B. One seeks to connect to the turning point a solution of type corresponding to λ_2 but now however let $\lambda_1 \equiv \lambda_2 \equiv \ldots \equiv \lambda_\ell$ but $\lambda_j \not\equiv \lambda_1$ for $x \not\equiv 0$ and $j > \ell$. In order to construct the appropriate comparison equation imagine A and B to be partioned into four parts by two straight lines, one line just below row ℓ and the other line just to the right of column ℓ. B is then separated into the sum of two parts

$$B \equiv B_1 + \hat{B}$$

where $\hat{B}$ is identically zero in the upper left partition but all other elements of $\hat{B}$ are identically the same as those of B. The comparison equation is

$$G' = (\rho A + B_1)G$$

and it is clear that with the above partitioning this equation splits into two separate equations. Consequently, $G(x)\, G^{-1}(\alpha)\hat{B}$ and $\hat{B}$ have the same placement of identically

zero elements. Moreover, although the upper left partition of $G(x)$ may not be known in closed form yet it is well defined as $\exp \rho \int_0^x \lambda_1(t)dt$ multiplied by the solution of a parameter free linear ordinary differential equation near an ordinary point. The analysis proceeds as in the previous section with the modified norm

$$|v| = \sum_{i=1}^{\ell} |v_i| + \rho^{\theta} \sum_{i=\ell+1}^{n} |v_i| \quad .$$

B. The Diagonalizable Leading Matrix

When the leading matrix is <u>not</u> diagonal it is natural to attempt to diagonalize it by a co-ordinate transformation. If w is the new dependent variable defined by

$$u = Pw$$

then the new equation is

$$w' = (\rho\, P^{-1}AP + P^{-1}BP - P^{-1}P')w$$

For the columns of P one chooses independent eigenvectors of A. The eigenvalues of A are presumed to be completely distinct for $x \neq 0$ and partially coalescent at $x = 0$. The matrix $P^{-1}AP$ has then the eigenvalues of A along the main diagonal and zeros elsewhere. These eigenvalues are certainly <u>continuous</u> at $x = 0$. Moreover, if ϕ is a column of P corresponding to the eigenvalue λ then the components of ϕ are proportional to the cofactors of <u>some</u> row in $A-\lambda I$. Hence the elements of P can certainly be chosen to be <u>continuous</u> at $x = 0$. For the moment let us presume that P^{-1} is <u>continuous</u> at $x = 0$. Then the new equation can be written as

$$w' = (\rho D + \hat{B})w$$

where D is diagonal with Lipschitz continuous elements and where the elements of $\hat{B}$ are integrable. The argument now proceeds as given in the preceding section with obvious alterations. For example, the asymptotic estimate of each exponential integral term now depends upon the degree of integrability of the integrand, i.e., for $q > -1$

$$\int_{x}^{x_o} t^q \exp \rho(x^k - t^k)dt = 0\left(\varepsilon^{\frac{1+q}{k}}\right)$$

for all x, including $x = 0$, in an appropriate sector. Also the θ to be employed for the modified norm is

$$\theta = \frac{1}{2} \text{Min}\left(\frac{1}{k_{ij}}\right) .$$

Let us now briefly return to the restrictions to be imposed upon $A(x)$ so that both $P(x)$ and $P^{-1}(x)$ are continuous at $x = 0$ where $P^{-1}AP$ is diagonal. Obviously, it is necessary that $A(0)$ have simple elementary divisors. More precisely, it is necessary that the derivative of lowest order at $x = 0$, say $A^{(\ell)}(0)$, which is not a scalar matrix have simple elementary divisors. (This form of the necessary condition is apparent from the trivial remark that P will diagonalize any scalar matrix). On the other hand, it is sufficient that the above defined $A^{(\ell)}(0)$ have distinct eigenvalues (since the columns of P, i.e., the eigenvalues

of A, will then certainly remain independent). Alternately, it is sufficient that $A(x)$ be normal (for then the normalized eigenvectors are orthogonal and hence cannot coalesce).

V. Weak Perturbations

The weak perturbation dispenses with the estimates for the exponential integrals. It is enough that a fundamental solution of the comparison equation be bounded as $\rho \longrightarrow \infty$ uniformly for x on a closed line from the turning point. The rapid convergence of the iterants of the integral equations is then provided by the portion of the integral kernel which corresponds to the weak perturbation. The weaker the perturbation, the more rapid the convergence for large ρ. To complete the picture one should delimit the class of comparison equations which satisfy the boundedness condition. A possible approach to the problem is the following. The classical demonstration of the asymptotic nature of the formal solutions first sets up a Volterra integral equation whose inhomogeneous term is a truncated formal solution. This integral equation is then solved by iteration for the variable limit of the integral within a sector but always a finite distance from the turning point. However, one can recognize the successive iterants to be the ubiquitous iterated exponential integrals. Furthermore, the inhomogeneous term admits a modified norm and finally the appropriate element of the kernel is asymptotically small. Hence by an application of the strong perturbation scheme one extends the far zone (where the formal expansions are asymptotic) down to an

infinitesmal distance (as $\rho \longrightarrow \infty$) from the turning point. But in the near zone a simple stretching of the variable results in an equation with coefficients uniformly bounded as $\rho \longrightarrow \infty$. Hence in the near zone the solution is a bounded function of the initial values. When the near and far zones meet at an intermediate point then a bounded comparison solution can be constructed .

VI. The Decoupling or Reduction Approach

The decoupling approach attempts to regard a system as a relatively weak coupling of components where the behavior of each component is either known or readily obtainable. From this one can usually deduce the behavior of the coupled system. For example, suppose

$$G_{11}' = \rho A_{11} G_{11}$$

and

$$G_{22}' = \rho A_{22} G_{22}$$

where A_{11}, A_{22} are square matrices and G_{11}, G_{22} are fundamental matrix solutions. Then a solution to

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \left\{ \rho \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \right\} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is

$$u_1 = g_1 + \int^x G_{11} B_{12} u_2$$

$$u_2 = g_2 + \int^x G_{22} B_{21} u_2 \quad .$$

If now $g \equiv g_1 \oplus g_2$ is a column of the composite $G \equiv G_{11} \oplus G_{22}$ and if the appropriate asymptotic behavior of G is known then the problem is ideally set up for the strong perturbation formalism.

As a second and final example, let us consider the decoupling due to non-coalescing eigenvalues (Langer [2]). The equation is as before

$$u' = (\rho A + B)u$$

where $A(x)$ has distinct eigenvalues for $x \neq 0$ but now one eigenvalue, say λ_1, remains distinct <u>at</u> the turning point $x = 0$. For simplicity λ_1 can be taken as identically zero. Let $[u_{(1)}]$ be a truncated formal solution corresponding to λ_1. The key fact is that $[u_{(1)}]$ is analytic in a full neighborhood of the turning point. This fact can be seen as follows. Since the eigenvalue remains distinct it certainly is analytic and consequently the eigenvector can be chosen to have analytic components. One augments this vector by additional analytic vectors to form a complete unitary set. Then one transforms u by this vector set regarded as a matrix. The leading matrix of the transformed equation will have only zeros in the first column. When a formal solution of type 1 is substituted into this transformed equation the equations which govern the successive terms reveal these successive terms to be analytic.

Since $[u_{(1)}]$ is an approximate solution even at the turning point hence it is not surprising that one can reduce the order of the original equation in the standard manner (Wasow [1, p. 28]). Namely, define P from the equation $P[u_{(1)}] = (\delta_{1i})$ and then change the dependent variable to w by the transformation

$$Pu = w \quad .$$

The equation for w is then

$$w' = \left\{P(\rho A + B)P^{-1} + P'P^{-1}\right\} w \equiv Fw \quad .$$

But from the definition of P, w has a solution asymptotic to (δ_{1i}). Consequently, the first column of F is asymptotic to zero so that the last $n-1$ components of w decouple asymptotically. Of course, as compared with the full system, the decoupled system has less transition lines and hence larger maximum sectors. Suppose then that a complete set of vector solutions of the decoupled equation is connected to the turning point in this larger sector. These vectors become solutions of the full equation when the appropriate first component of w is appended. These first components are readily determined. However, they are then connected to the turning point in a maximum sector determined by the <u>full</u> equation.

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